A Perturbation Approach to Filtering Hidden States

Ivana Komunjer and Natalia Sizova

UCSD and Rice University

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Motivation:

Dynamic stochastic choice models are a signature tool in Economics:

- growth models in macroeconomics
- portfolio choice models in finance

Hidden states (more generally, incomplete information over the model states) are an increasingly common feature in these models.

Motivation:

With incomplete information over states, optimal decision rules depend on the decision maker's posterior distribution over the hidden states given the observed history

- control theory: Striebel (1965), Rhenius (1974), Yuschkevich (1976), Bertsekas and Shreve (1978)
- economics: Hansen and Sargent (2005, 2007)

Consequence:

- models with incomplete information have infinite dimensional state vectors
- solving and analyzing such models is a challenge

This Paper:

We use the perturbation method to derive an approximate filter for models in which the hidden state follows a linear transition equation, but the observation equation is possibly nonlinear.

Leading example: dynamic stochastic choice models with stochastic volatility.

Stochastic Growth Model with Stochastic Volatility

A simple neoclassical growth model with stochastic volatility (Caldara, Fernández-Villaverde, Guerrń)

A representative consumer chooses $c_0, c_1, \ldots$ so as to maximize

$$E\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right]$$

subject to:

$$\begin{aligned}
k_{t+1} &= (\exp z_t) k_t^{\alpha} + (1-\delta) k_t - c_t \\
z_{t+1} &= \rho z_t + (\exp \sigma_t) \epsilon_{t+1} \\
\sigma_{t+1} &= (1-\lambda)\bar{\sigma} + \lambda \sigma_t + \eta \omega_{t+1}
\end{aligned}$$

Stochastic Growth Model with Stochastic Volatility

States of the model:

$$\underbrace{k_t, z_t}_{\text{observed states}}, \quad \underbrace{\sigma_t}_{\text{hidden state}}$$

Decision sequence:

- $t = 0$, the initial values k_0, z_0 and the density p_0 for σ_0 are known, and $c_0(k_0, z_0, p_0)$ is chosen
- system moves stochastically from (k_t, z_t, σ_t) to $(k_{t+1}, z_{t+1}, \sigma_{t+1})$
- observed k_{t+1}, z_{t+1} are added to the time $t+1$ information vector:

$$i_{t+1} = (k_0, z_0, p_0, k_1, z_1, \ldots, k_{t+1}, z_{t+1})$$

- the decision maker chooses $c_{t+1}(i_{t+1})$

Stochastic Growth Model with Stochastic Volatility

Even though the state (k_t, z_t, σ_t) follows a Markov process, the optimal policy c_t is not Markov. It depends on the entire information vector i_t.

Problem: we can no longer limit ourselves to Markov policies only

Solution: reformulate the decision problem as a problem with perfectly observed state that has the Markov property

Stochastic Growth Model with Stochastic Volatility

Denote by

$$p_t(\sigma) \text{ the conditional density of } \sigma_t \text{ given } i_t$$

Then, p_t satisfies a recursive relation of the form:

$$p_{t+1} = \phi(z_{t+1}, z_t, p_t)$$

where the function ϕ is known.

Specifically:

$$p_{t+1}(\sigma_{t+1}) = \frac{\int_{\mathbb{R}} \frac{1}{\eta} p_\omega \left(\frac{\sigma_{1+1} - \bar{\sigma}}{\eta} - \lambda \frac{\sigma - \bar{\sigma}}{\eta} \right) \frac{1}{\exp \sigma} p_\epsilon \left(\frac{z_{t+1} - \rho z_t}{\exp \sigma} \right) p_t(\sigma) d\sigma}{\int_{\mathbb{R}} \frac{1}{\exp \sigma} p_\epsilon \left(\frac{z_{t+1} - \rho z_t}{\exp \sigma} \right) p_t(\sigma) d\sigma},$$

Stochastic Growth Model with Stochastic Volatility

Value function:

$$v(k_0, z_0, p_0) = \sup E\left[\sum_{t=0}^{\infty} \beta^t u(c_t) \Big| k_0, z_0, p_0\right]$$

Bellman equation:

$$v(k, z, p) = \sup_{c \in C} \left\{ u(c) + \beta \int_{\mathbb{R}^2} v\left(k', z', \phi(z', z, p)\right) p_\epsilon(\epsilon') p(\sigma) d\epsilon' d\sigma \right\},$$

where

$$\begin{aligned} k' &= (\exp z) k^\alpha + (1 - \delta) k - c \\ z' &= \rho z + (\exp \sigma) \epsilon' \end{aligned}$$

Stochastic Growth Model with Stochastic Volatility

Optimal policy:

$$c(k, z, p)$$

Euler equation:

$$u'(c_t) = \beta \int_{\mathbb{R}^2} \left[\alpha (\exp z_{t+1}) k_{t+1}^{\alpha-1} + (1-\delta) \right] u'(c_{t+1}) p_{\epsilon}(\epsilon_{t+1}) p_t(\sigma_t) d\epsilon_{t+1} d\sigma_t,$$

Problem: the state p_t is infinite dimensional!

Our solution: approximate p_t by a finite number of sufficient statistics using the perturbation method

Main Result

Transition equation:

$$z_{t+1} = \rho z_t + (\exp \sigma_t)\epsilon_{t+1}$$
$$\sigma_{t+1} = (1-\lambda)\bar{\sigma} + \lambda \sigma_t + \eta \omega_{t+1}$$

with ϵ_t is iid with a density f_ϵ on $\mathbb{R}$ satisfying $E(\epsilon_t) = 0$ and $E(\epsilon_t^2) = 1$, ω_t is iid Gaussian with $E(\omega_t) = 0$ and $E(\omega_t^2) = 1$, ϵ_t and ω_t are independent.

Main Result

Perturbation Approach:

$$z_{t+1} = \rho z_t + \Lambda(\exp \sigma_t)\epsilon_{t+1}$$
$$\sigma_{t+1} = (1-\lambda)\bar{\sigma} + \lambda\sigma_t + \Lambda\eta\omega_{t+1}$$

Idea: approximate p_t around $\Lambda = 0$ (noninformative case, $\sigma_t = \bar{\sigma}$)

Literature:

- control: Fleming (1971), Bensoussan (1988)
- economics: Judd (1996)

Main Result

First-order approximation for p_t

$$p_t(\sigma_t) = \frac{\exp\left(-\frac{(\sigma_t-\bar{\sigma})^2}{2\eta^2\sigma_0^2}\right)}{\sqrt{2\pi\eta^2\sigma_0^2}} \left\{1 + A_{1,t}\left(\sigma_t - \bar{\sigma}\right) + o(\|\eta\|, \|\sigma_t - \bar{\sigma}\|)\right\},$$

where $\sigma_0^2 = (1-\lambda^2)^{-1}$, and

$$A_{1,t+1} = \lambda\left[A_{1,t} - \psi_1\left(\frac{z_{t+1} - \rho z_t}{\exp\bar{\sigma}}\right)\right],$$

with $\psi_1(y) \equiv 1 + y f_\varepsilon'(y)/f_\varepsilon(y)$.

One sufficient statistic $A_{1,t}$ in the first order approximation. Higher orders are possible; more sufficient statistics appear.

How to compare different volatility filters?

Since

$$\exp(2\sigma_t) = E\left[(z_{t+1} - \rho z_t)^2 | z_t\right],$$

we have

$$E\left[(z_{t+1} - \rho z_t)^2 - \exp(2\sigma_t)\right] = 0.$$

So we can compare the accuracy of different volatility filters $\hat{\sigma}_t$ by looking at MSE loss:

$$MSE(\hat{\sigma}) = E\left[(z_{t+1} - \rho z_t)^2 - \exp(2\hat{\sigma}_t)^2\right].$$

How to compare different volatility filters?

In simulations, the true value of σ_t is known, so then we can also look at how far the filtered volatility is from the true volatility by calculating

$$L(\hat{\sigma}) = E\left[(\sigma_t - \hat{\sigma}_t)^2\right].$$

Some Simulation results

$T = 500$, we compare the following filters: particle filter ($M = 100000$) $\hat{\sigma}_P$, with approximation filters: first order $\hat{\sigma}_1$, second order $\hat{\sigma}_2$, third order $\hat{\sigma}_3$

	MSE	L	RMSE
σ	142.4261	0	1
$\hat{\sigma}_P$	143.4061	0.0185	1.0069
$\hat{\sigma}_1$	144.2345	0.0194	1.0127
$\hat{\sigma}_2$	144.2345	0.0194	1.0127
$\hat{\sigma}_3$	143.0395	0.0186	**1.0043**

Conclusion

In this paper, we use regular perturbations à la Fleming (1971), Bensoussan (1988), and Judd (1996), to derive an approximate filter for the conditional distribution of the hidden state.

The key features of the filter are:

1. it requires linear dynamics of the hidden state, but allows for nonlinear observation equation
2. it approximates the information in the hidden state distributions by a finite number of sufficient statistics whose dynamics are again linear
3. implementation of the filter is straightforward unlike in particle filtering methods